Homology Theory for Operator Algebras: Traditional and Quantized Aspects.

Helemskii A. Ya.

Faculty of Mechanics and Mathematics
Moscow State University
Moscow 119899GSP
RUSSIA

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Let A be an operator algebra, that is a closed subalgebra of the algebra $\mathcal{B}(H)$ of all bounded operators on a Hilbert space H. We shall discuss some homological conditions on A that garantee the best possible, in some reasonable sense, structure of A. As a sample, we take the classical Wedderburn theorem that in the modern language can be formulated as follows.

Let A be an algebra of operators acting on a finite-dimensional linear space H. Then there are decompositions $H = \bigoplus\{H_\nu : \nu \in \Lambda\}$ and $H_\nu = H'_\nu \bigotimes H''_\nu$ such that A consists of all operators such that all H_ν are their invariant subspaces and such that their respective restrictions have the form $a \otimes 1$, where a acts on H'_ν.

With the establishment of functional analysis some people and notably von Neumann began to be interested in possible functional-analytic generalizations of this theorem. This interest was one of the main impulses that led von Neumann to discover what we call now "von Neumann algebras".

The desired property of an algebra which could participate in the proper generalization of Wedderburn theorem is evidently as follows. Let now H be an arbitrary Hilbert space. We call an operator algebra on H *Wedderburn operator algebra* if there exist decompositions into the Hilbert sum and

Hilbert tensor products, similar to the indicated before (now with, generally speaking, infinite-dimensional Hilbert spaces and an arbitrary cardinality of Λ) such that A consists of all bounded operators of the indicated form.

It is known that at the beginning of 30-es von Neumann supposed that Wedderburn algebras are just von Neumann algebra with discrete (i.e., isomorphic to $l_\infty(\cdot)$) center. But in 1935, jointly with Murrey, he found an example of an algebra with (even) scalar center which is not Wedderburn (and, in fact, behaves very differently from Wedderburn algebras). Now, in retrospective, we know that it was one of major mathematical discoveries of this century.

Nevertheless, the problem of characterizing Wedderburn algebras remained. In the course of this century,at first in pure algebra and then in functional ansalysis, the tradition, now quite time-honoured, was gradually established: to study rings and algebras by what is called know homological methods. In particular, the homological property of projectivity came to the forefront. The most known theorem of that kind, again giving us a sample, can be formulated as follows. *Let A be an algebra with projective modules and bimodules. Then it is finite-dimensional and semi-simple.* (In particular, for a selfadjoint operator algebra this implies that it is Wedderburn).

But how should we distinguish general, not necessarily finite-dimensional, Wedderburn algebras? In a paper of 1994, we suggested to consider the so-called spatially projective algebras (see below). It was shown that such a property distinguishes a rather wide class of Wedderburn algebras but not all of them. Namely, we obtain only the so-called *essentially finite* algebras, that is such Wedderburn algebras that in the indicated decomposition of H_ν we have, for any $\nu \in \Lambda$, that at least one of respective Hilbert factors is finite-dimensional.

Thus this attempt to characterize Wedderburn algebras, which have used traditional concepts of Banach homology, had a partial success. Meanwhile during last 20 years the new rapidly developping branch of functional analysis appeared, what is now called quantized functional analysis (or, more prosaically, theory of operator spaces). The leading figures on this area, notably Effros, Paulsen and Blecher, after "quantizing" Banach spaces, began to quantize algebras, modules and eventually homology. In particular, Paulsen suggested about 1996 the quantized version of a traditional notion of a Banach projective module. From this one immedeately can derive the respective quantized version of a notion of spatial projectivity. And this happened to be quite what we needed: at the end of last (1998) year we have proved that "quantum spatially projective" operator algebras are exactly Wedderburn algebras, without any additional conditions. That was, in a sense, the end of the story.

Now, after this outline, we shall give precise definitions and formulations.

Let A be a (so far arbitrary) Banach algebra. In what follows "A-module" always means a left Banach module over A. Recall that a continuous linear operator $\varphi : X \to Y$ between two A-modules is called a morphism if $\varphi(a \cdot x) = a \cdot \varphi(x)$ for all $a \in A, x \in X$. A surjective morphism is called *admissible* if it has a right inverse continuous linear operator.

Definition. (1970). An A-module P is called *projective* or, to be precise, *traditionally projective* if, for any admissible surjective morphism $\sigma : X \to Y$ between A-modules and for any morphism $\varphi : P \in Y$ there exists a morphism $\psi : P \to X$ such that $\sigma\psi = \varphi$. An operator algebra A on a Hilbert space H is called *spatially projective*, if H, as the spatial A-module (i.e. with the outer multiplication $a \cdot x := a(x)$) is projective.

To present the quantum version of this concept, we need several basic definitions of quantized functional analysis. First of all, for a linear space E, a norm in $M_\infty \otimes E = M_\infty(E)$, where M_∞ is the algebra of infinite (to the right and to below) matrixes with the finite number of non-zero entries, is called a *matrix-norm* in E. A matrix-norm in a Banach space E is called a *quantization* if it is obtained with the following procedure. At first we take an (arbitrary) isometrical embedding of E into $\mathcal{B}(H)$ for some H, then take the respective injective map of $M_\infty \otimes E$ into $M_\infty \otimes \mathcal{B}(H))$ and finally, identifying the latter space with the respective subspace in $\mathcal{B}(l_2 \otimes H)$, we supply $M_\infty \otimes E)$ with the induced norm. (We emphasize that different initial isometrical embeddings provide, generally speaking, different quantizations of the same Banach space). A Banach space, endowed with a quantization, is called a *quantum Banach space.*

Any operator algebra has the so-called *standard* quantization, provided by its natural embedding into respective $\mathcal{B}(H)$. As to many useful quantizations of a given Hilbert space H, we shall need here only the following, the so-called *column* quantization. It is provided by the embedding H into $\mathcal{B}(H)$, taking $x \in H$ to the rank-one operator $y \mapsto (y, e)x$ where e is a fixed normed vector in H.

An operator $\varphi : E \to F$ between two quantum spaces is called *completely bounded* if the operator $1 \otimes \varphi : M_\infty \otimes E \to M_\infty \otimes F$ is bounded. A bilinear operator $\varphi : E \times F \to G$ between three quantum spaces is called *completely bounded* if the bilinear operator from $M_\infty \otimes E) \times M_\infty \otimes$ to $M_\infty \otimes G$, well-defined by $(a \otimes x, b \otimes y) \mapsto ab \otimes \varphi(x, y)$, is bounded. A Banach algebra, endowed with a quantization, is called a *quantum algebra* if its multiplication is, as a bilinear operator, completely bounded. Finally, if a module over a quantum algebra is endowed with a quantization, it is called a *quantum module* if its outer multiplication is, as a bilinear operator, completely bounded.

It is easy to check that the spatial module over an operator algebra with the standard quantization is a quantum module if we endow it with the column quantization. This quantum module will be called *standard spatial module.*

Let A be a quantum algebra. A completely bounded (as an operator) morphism between two quantum A-modules is called *quantum admissible* if it has a right inverse completely bounded operator.

Definition. A quantum A-module P is called *quantum projective* if, for any quantum admissible surjective morphism $\sigma : X \to Y$ between quantum A-modules and for any completely bounded morphism $\varphi : P \in Y$ there exists a completely bounded morphism $\psi : P \to X$ such that $\sigma\psi = \varphi$. An operator algebra, endowed with the standard quantization, is called *quantum spatially projective*, if its standard spatial module is projective.

Now we are able to formulate our main result.

Theorem. *Let A be a von Neumann algebra. Then*

(i) A is traditionally spatially projective iff it is Wedderburn and essentially finite.

(ii) A is quantum spatially projective iff it is (just) Wedderburn.